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AN ENVELOPE THEOREM AND NECESSARY CONDITIONS FOR A PROBLEM OF MAYER WITH VARIABLE END POINTS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
MOFFATT GRIER BOYCE

L. C.

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This dissertation is one of a group of five submitted during the year 1930 to the Department of Mathematics of the University of Chicago and concerned with topics in the theory of the Calculus of Variations. The group as a whole has been published in book form by the University of Chicago Press under the title *Contributions to the Calculus of Variations, 1930.*

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INTRODUCTION

The general problem of Mayer with variable end points as proposed by Bliss[1] is that of finding in a class of arcs

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2 ; i = 1,2,\cdots,n)$$

which satisfy a system of differential equations

$$\varphi_\alpha(x, y_1, \cdots, y_m, y_1', \cdots, y_m') = 0 \qquad (\alpha = 1,2,\cdots,m<n)$$

one that minimizes the first of a set of functions

$$f_\sigma(x_1, y_1(x_1), \cdots, y_m(x_1), x_2, y_1(x_2), \cdots, y_m(x_2)) \quad (\sigma = 0,1,\cdots,r \leqq 2n+1)$$

while making the rest to vanish. A multiplier rule with transversality conditions has been deduced by Bliss[1]. He also showed this problem and a general problem studied by Bolza[2] to be equivalent in the sense that the multiplier rule for each is derivable from that of the other. T. F. Cope[3] has obtained by boundary value methods an analogue of Jacobi's condition, which however bears little similarity to the corresponding results of this paper, where more nearly the classical calculus of variations mode of attack is adopted. These classical methods have been used by E. L. Mackie[4] to obtain a Jacobi condition for a less general problem than the one studied here.

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[1] The problem of Mayer with variable end points, Transactions of the American Mathematical Society, vol. 19 (1918), p. 305; hereafter referred to as "Trans. XIX".

[2] Über den "Anormalen Fall" beim Lagrangeschen und Mayerschen Problem mit gemischten Bedingungen und variablen Endpunkten, Mathematische Annalen, vol. 74 (1913), p. 430.

[3] An analogue of Jacobi's condition for the problem of Mayer with variable end points, Dissertation, University of Chicago, 1927.

[4] The Jacobi condition for a problem of Mayer with variable end points, Dissertation, University of Chicago, 1927; also a paper by the same title, Annals of Mathematics, series 2, vol. 30 (1929), p. 393.

In this paper the generality of the problem is restricted to the extent of assuming the first end point to be fixed. Section 1 consists of a definite formulation of the problem and preliminary notions. In Sections 2 and 3 standard methods yield a multiplier rule in the Du Bois-Reymond form and a 2n-parameter family of extremals. Certain needed theorems concerning normal arcs are proved in Section 4. Section 5 contains analogues to the Weierstrass and Legendre conditions. This section is non-essential to the sequel. The remaining three sections, dealing with the development of an envelope theory, are believed to constitute the part of chief interest. An analogue of Kneser's envelope theorem is given in Section 6, and in Section 7 a Jacobi condition is shown to follow as a consequence. Incidentally, it was found possible to obtain the Jacobi condition with weaker normality hypotheses than those usually made for the Lagrange problem. Section 8 contains two methods for selecting from the 2n-parameter family of extremals a one-parameter family having an envelope with the desired properties. The two methods are shown to give the same focal points.

The procedure was suggested by the dissertation of Mackie, Lectures by Bliss[1], Bolza's Vorlesungen uber Variationsrechnung, and a paper by G. A. Larew[2].

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[1]Mimeographed notes, The problem of Lagrange in the calculus of variations, Summer 1925. This paper will be referred to as "Lectures".

[2]Necessary conditions in the problems of Mayer in the calculus of variations, Transactions of the American Mathematical Society, vol. 20 (1919), pp. 1-22; referred to as "Trans. XX".

1. FORMULATION OF THE PROBLEM

A set of elements or functions will usually be denoted by a single element, without a subscript when representing the arguments of a function but otherwise with a subscript to indicate the range. For example, in $\varphi_\alpha(x,y,y')$, y stands for $y_1, y_2, \cdots, y_m$ and φ_α for $\varphi_1, \cdots, \varphi_m$. Repeated subscripts will indicate sums as in tensor analysis. For example, $\lambda_\alpha\varphi_\alpha$ will mean $\lambda_1\varphi_1 + \cdots + \lambda_m\varphi_m$. The subscripts used will always have the following ranges:

$$i,k = 1,\ldots,n; \quad \alpha = 1,\ldots,m<n; \quad \tau = m+1,\ldots,n;$$

$$\rho = 1,\ldots,r\leqq n; \quad \sigma = 0,1,\ldots,r; \quad \pi = 1,\ldots,p;$$

$$\omega = 1,\ldots,r+n; \quad g = 1,\ldots,n-r+1; \quad h = n-r+2,\ldots,n.$$

Other literal subscripts will denote partial derivatives, and primes, derivatives with respect to x. A point with coordinates $x_1, y_{i1}(x_1)$ will be referred to as the point 1 and will have its coordinates denoted by x_1, y_{i1} or (x_1,y_1). Corresponding notation will be used for functions η_i and points 2, 3, etc.

By definition let an admissible arc be an arc

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2) \tag{1.1}$$

of class[1] D' lying in an open region R of (x,y,y')-space and satisfying the system of differential equations

$$\varphi_\alpha(x,y,y') = 0. \tag{1.2}$$

The problem to be studied then is that of finding in the class

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[1] The word "class" is used here as by Bolza, Vorlesungen über Variationsrechnung, pp. 13,63.

of admissible arcs one which has a given initial point (x_1, y_1) and whose end point (x_2, y_2) minimizes the first of the set of functions $f_\sigma(x_2, y_2)$ while making the rest to vanish.

We shall direct attention to a particular arc E_{12} with equations (1.1) and investigate the properties it must have if it is to be a minimizing arc. The analysis will be based on the following hypotheses:

1. The arc E_{12} is an admissible arc.

2. The first end point of E_{12} is the given point 1, and the second satisfies the equations $f_\rho(x_2, y_2) = 0$.

3. The functions φ_α and f_σ are of class C''' for values of their arguments near those defining E_{12}.

4. The m by n matrix $\| \partial\varphi_\alpha / \partial y'_k \|$ is of rank m at every point of E_{12}.

5. The r+1 by n+1 matrix $\| \partial f_\sigma / \partial x_2 \;\; \partial f_\sigma / \partial y_{i2} \|$ is of rank r+1 when the arguments are those of E_{12}.

When there is danger of confusing the arguments of E_{12} with those of other arcs, bars will be used over the former: as, $\bar{y}_i$, $\bar{x}_2$.

It has been shown by Bliss[1] that n-m functions $\varphi_\tau(x, y, y')$ can always be adjoined to the set φ_α in such a way that the extended set φ_i has a determinant $|\partial\varphi_i / \partial y'_k|$ different from zero along E_{12} and has the continuity properties of Hypothesis 3 between the corners of E_{12}. The equations

(1.3) $$\varphi_i(x, y, y') - z_i = 0$$

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[1] Trans. XIX, p. 312.

then determine for E_{12} new variables $z_\alpha \equiv 0, z_\tau$. Admissible arcs for this extended set of differential equations will be defined as before, with $z_\alpha \equiv 0, z_\tau$ adjoined to the set y_i and equations (1.3) replacing (1.2).

A set of <u>admissible variations along E_{12}</u> for the extended system of differential equations will consist of any constant ξ_2 and set of functions $\eta_k(x)$ of class D', $\xi_\alpha \equiv 0, \xi_\tau(x)$ of class D which satisfies the equations

$$\frac{\partial \varphi_i}{\partial y_k}\eta_k + \frac{\partial \varphi_i}{\partial y_k'}\eta_k' - \xi_i = 0, \tag{1.4}$$

where the arguments are those of E_{12}. For the unextended system the ξ's will be dropped and α will replace i in (1.4).

If a one-parameter family of admissible arcs

$$y_i = y_i(x,a),\ z_i = z_i(x,a)\quad [x_1 \leqq x \leqq x_2(a);\ -\delta < a < \delta] \tag{1.5}$$

of class C' in a contains E_{12} for $a = 0$, then the functions

$$\xi_2 = x_{2a}(0),\ \eta_i(x) = y_{ia}(x,0),\ \xi_i(x) = z_{ia}(x,0)$$

form a set of admissible variations along E_{12}. Conversely, by Lemma 2 below, every set of admissible variations along E_{12} has a family (1.5) related to it as above and containing E_{12} for $a = 0$. If the family (1.5) also satisfies the end relations $f_\rho = 0$, then the variations of the family along E_{12} give the value zero to all except the first of the functions

$$F_\sigma(\xi,\eta) = \left(\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}}y_{i2}'\right)\xi_2 + \frac{\partial f_\sigma}{\partial y_{i2}}\eta_{i2},$$

where the arguments are those of E_{12}.

The following lemmas[1] are consequences of the existence

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[1] Cf. Bliss, Lectures, p.4; Trans.XIX, p.308.

theorems[2] for implicit functions and differential equations.

LEMMA 1. The equations (1.4) determine uniquely for E_{12} a set of solutions η_i of class D′ in a neighborhood of the values $x_1 \leqq x \leqq x_2$ when functions ξ_i of class D in this neighborhood and the initial values of the η's at a single point are given.

LEMMA 2. If p sets of admissible variations $\xi_2^\pi, \eta_i^\pi, \xi_i^\pi$ are given, then there exists a p-parameter family of admissible arcs

$$y_i = y_i(x,a) \qquad [x_1 \leqq x \leqq x_2(a);\ a=(a_1,\cdots,a_p)]$$

containing E_{12} for $a_\pi = 0$ and having $\xi_2^\pi, \eta_i^\pi, \xi_i^\pi$ as variations along E_{12}. The functions $y_i, \partial y_i/\partial a_\pi$ are continuous for all values (x,a) near those defining E_{12}, and the functions $\partial y_i'/\partial a_\pi$ have the same property except possibly for the values of x defining the corners of E_{12}.

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[2] Bolza, loc. cit., p. 166ff; Bliss, Solutions of differential equations as functions of the constants of integration, Bulletin of the American Mathematical Society, vol. 25(1918), p.15. For an extension to arcs of class D′, see Larew, Trans. XX, p.4.

2. THE MULTIPLIER RULE

The multiplier rule given by Bliss for the Mayer[1] problem is for minimizing arcs of class C''. His proof for the Lagrange[2] problem, however, assumes continuity of class D' only, which is the class desired here. The methods of these proofs are easily adapted to the present case.

From the implicit function theorems and Lemma 2 of the preceding section it can be proved that if a system of $n+r+1$ sets of variations $\xi_2^\gamma, \eta_i^\gamma$ exists such that the determinant $|\eta_{i1}^\gamma \; F_\sigma(\xi^\gamma \eta^\gamma)|$ is different from zero then E_{12} can not make f_0 a minimum. This implies the

LEMMA. *If E_{12} makes f_0 a minimum then there must exist constants g_i, ℓ_σ such that the ℓ's are not all zero and the sum*

$$g_i \eta_{i1} + \ell_\sigma F_\sigma \tag{2.1}$$

vanishes for every set of admissible variations.

Now to obtain the multiplier rule let n functions $\lambda_i(x)$, continuous except possibly at corners of E_{12}, be chosen to satisfy the equations

$$\Omega_{y_i'} = \int_{x_1}^{x} \Omega_{y_i} dx + c_i , \tag{2.2}$$

where Ω denotes the sum $\lambda_i \varphi_i$, the c_i are arbitrary constants, and the arguments are those of E_{12}. That this can be done may be shown by introducing new variables

$$v_i = \Omega_{y_i'} . \tag{2.3}$$

Then the equations (2.2) are equivalent to the differential

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[1] Bliss, Trans. XIX, pp. 308-311.
[2] Bliss, Lectures, pp. 8-12, 18-21.

equations and initial conditions

$$\frac{dv_i}{dx} = \Omega_{y_i} = A_{ik} v_k, \quad v_i(x_1) = c_i, \tag{2.4}$$

where the A_{ik} are functions of x of class D determined by using the values of λ_k obtained from solving the equations (2.3). Equations (2.4) have unique solutions $v_i(x)$ of class D' on $x_1 x_2$ which, when substituted back into equations (2.3), determine uniquely in terms of the arbitrary c's the functions $\lambda_i(x)$, continuous on $x_1 x_2$ except possibly at the corners of E_{12}. From equations (1.4) and (2.2) it follows that

$$\begin{aligned} \int_{x_1}^{x_2} \lambda_i \xi_i \, dx &= \int_{x_1}^{x_2} \left(\Omega_{y_i} \eta_i + \Omega_{y_i'} \eta_i' \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{d}{dx} \eta_i \int_{x_1}^{x} \Omega_{y_i} dx + c_i \eta_i' \right) dx \\ &= \eta_i(x_2) \left(\int_{x_1}^{x_2} \Omega_{y_i} dx + c_i \right) - c_i \eta_i(x_1). \end{aligned} \tag{2.5}$$

If E_{12} is a minimizing arc, the sum (2.1) may be added to the above result, giving

$$\begin{aligned} \int_{x_1}^{x_2} \lambda_i \xi_i \, dx &= (g_i - c_i) \eta_{i1} + \ell_\sigma \left(\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}} y_{i2}' \right) \xi_2 \\ &\quad + \left(\ell_\sigma \frac{\partial f_\sigma}{\partial y_{i2}} + \int_{x_1}^{x_2} \Omega_{y_i} dx + c_i \right) \eta_{i2}, \end{aligned} \tag{2.6}$$

which must hold for all admissible variations and therefore by Lemma 1 of Section 1 for arbitrarily selected values ξ_2, η_{i1}, ξ_i. Choose c_i so as to make the coefficients of η_{i2} vanish. From (2.2) we have

$$(2.7)\qquad \Omega_{y_i'}(x_1) = c_i\,, \quad \Omega_{y_i'}(x_2) = \int_{x_1}^{x_2} \Omega_{y_i}\,dx + c_i\,.$$

It follows then that the last n-m λ's vanish identically and that

$$(2.8)\qquad g_i - \Omega_{y_i'}(x_1) = 0\,,$$

$$(2.9)\qquad \ell_\sigma\Big(\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}}\,y_{i2}'\Big) = 0, \quad \ell_\sigma\frac{\partial f_\sigma}{\partial y_{i2}} + \Omega_{y_i'}(x_2) = 0\,,$$

when the arguments are those of E_{12}. The m functions λ_α do not vanish identically, since then $\ell_\sigma\frac{\partial f_\sigma}{\partial x_2}$, $\ell_\sigma\frac{\partial f_\sigma}{\partial y_{i2}}$ would all vanish in contradiction of Hypothesis 5. Constants g_i satisfying (2.8) exist for any arc. Hence, omitting them, we have the multiplier rule:

THEOREM. *If E_{12} is a minimizing arc, then there exist n constants c_i and m functions $\lambda_\alpha(x)$ not identically zero on x_1x_2 and continuous except possibly at corners of E_{12}, such that the equations*

$$(2.10)\qquad \Omega_{y_i'} = \int_{x_1}^{x} \Omega_{y_i}\,dx + c_i$$

are satisfied at every point of E_{12}. Also there exist r+1 constants ℓ_σ not all zero, such that equations (2.9) are satisfied by the end values of E_{12}.

COROLLARY 1. *At every point of E_{12} not a corner the equations*

$$(2.11)\qquad \frac{d}{dx}\,\Omega_{y_i'} = \Omega_{y_i}\,, \quad \varphi_\alpha = 0$$

are satisfied.

COROLLARY 2. *At every corner of E_{12} the conditions*

$$(2.12)\qquad \Omega_{y_i'}(x, y, y'(x-0), \lambda(x-0)) = \Omega_{y_i'}(x, y, y'(x+0), \lambda(x+0))$$

must hold.

COROLLARY 3. Near a point of E_{12} which is not a corner and at which the determinant

$$R = \begin{vmatrix} \Omega_{y_i' y_k'} & \partial\varphi_\alpha/\partial y_i' \\ \partial\varphi_\alpha/\partial y_k' & 0 \end{vmatrix}$$

is different from zero, the arc E_{12} is of class C'' and the multipliers are of class C'.

COROLLARY 4. A sub-arc E_{32} of E_{12}, $x_1 < x_3 < x_2$, satisfies the multiplier rule on $x_3 x_2$ with the same multipliers λ_α, l_σ that belong to E_{12}. The c_i for E_{32} are equal to those for E_{12} plus $\int_{x_1}^{x_3} \Omega_{y_i} dx$.

The first two and fourth corollaries are immediate, and the third follows in the manner indicated by Bliss[1] for the Lagrange problem.

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[1] Lectures, p. 11.

3. THE 2n-PARAMETER FAMILY OF EXTREMALS

The theorem of this section follows in the same way as for the Lagrange problem[1]. The principal details of the proof will be given for purposes of later reference.

An admissible arc

$$y_i = y_i(x) \qquad (x_1 \leqq x \leqq x_2)$$

will be called an extremal if there exists a set of multipliers $\lambda_\alpha(x)$ with which it satisfies the Euler-Lagrange equations (2.11) and if the derivatives y_i', y_i'', λ_α' are continuous on $x_1 x_2$.

Let E_{12} be an extremal along which the determinant R of page 8 is different from zero. Then in a neighborhood of E_{12} the equations

$$\Omega_{y_i'}(x, y, y', \lambda) = v_i, \quad \varphi_\alpha(x, y, y') = 0 \tag{3.1}$$

have solutions

$$y_i' = \psi_i(x, y, v), \quad \lambda_\alpha = \pi_\alpha(x, y, v) \tag{3.2}$$

which, from the continuity properties of the φ's, have continuous partial derivatives of the first and second orders.

The system of equations (2.11) is now equivalent to the system in normal form

$$\frac{dy_i}{dx} = \psi_i(x, y, v), \quad \frac{dv_i}{dx} = \Omega_{y_i}(x, y, \psi(x, y, v), \pi(x, y, v)). \tag{3.3}$$

Through every initial element (x_0, y_0, v_0) in a neighborhood of

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[1] Cf. Bliss, Lectures, p. 13; Bolza, loc. cit., p. 590.

any set of values (x,y,v) on E_{12} there passes a unique solution

(3.4) $$y_i = y_i(x, x_0, y_0, v_0), \quad v_i = v_i(x, x_0, y_0, v_0)$$

of equations (3.3) for which the functions y_i, y_i', v_i, v_i' have continuous partial derivatives of the first and second orders. Since

$$y_{i0} = y_i(x_0, x_0, y_0, v_0), \quad v_{i0} = v_i(x_0, x_0, y_0, v_0),$$

it follows that the determinant

$$\begin{vmatrix} \frac{\partial}{\partial y_{k0}} y_i(x, x_0, y_0, v_0) & \frac{\partial}{\partial y_{k0}} v_i(x, x_0, y_0, v_0) \\ \frac{\partial}{\partial v_{k0}} y_i(x, x_0, y_0, v_0) & \frac{\partial}{\partial v_{k0}} v_i(x, x_0, y_0, v_0) \end{vmatrix}$$

has the value 1 for $x=x_0$. Hence, if x_0 is fixed, the 2n remaining parameters are independent. Thus, on substituting (3.4) in the functions π_α of (3.2) and on omitting the fixed value x_0, we have the theorem:

THEOREM. *If E_{12} is an extremal along which R does not vanish, then the solutions of the Euler-Lagrange equations* (2.11) *in a neighborhood of E_{12} constitute a 2n-parameter family of extremals and multipliers*

(3.5) $$y_i = y_i(x, y_0, v_0), \quad \lambda_\alpha = \lambda_\alpha(x, y_0, v_0)$$

containing E_{12} for special values $(y_0, v_0) = (\bar{y}_0, \bar{v}_0)$. The functions $y_i, y_i', \lambda_\alpha, \lambda_\alpha'$ have continuous partial derivatives of the first and second orders for their arguments in a neighborhood of those defining E_{12}.

To determine the 2n parameters in the family of extremals, the r+1 ℓ's in the transversality conditions, and x_1 and x_2 we have the n+1 coordinates of the first end point, r conditions $f_\rho = 0$ on the second end point, and n+1 transversality conditions (2.9). There are thus 2n+r+2 conditions on the

$2n+r+3$ quantities. However, only the ratios of the λ_α, l_σ are essential, and in the next section it is shown that for normal arcs l_0 may be taken equal to unity.

4. NORMAL ADMISSIBLE ARCS

An admissible arc E_{32} will be called normal[1] if there exist for it n+r sets of admissible variations $\xi_2^\omega, \eta_\lambda^\omega$ such that the determinant

$$\left| \eta_\lambda^\omega(x_3) \quad F_\rho(\xi_2^\omega \eta^\omega) \right| \tag{4.1}$$

is different from zero when the values of F_ρ are those for E_{32} at the point 2. If E_{32} is normal, then in every neighborhood of E_{32} there is an infinity of admissible arcs passing through the point 3 and satisfying the equations $f_\rho = 0$.[2] If E_{32} is not normal there always exists a set of multipliers $\ell_\sigma, \lambda_\alpha$ with $\ell_0 = 0$ with which E_{32} satisfies the multiplier rule. For, if the determinant (4.1) vanishes for every system of n+r sets of variations, then there exist constants g_λ, ℓ_ρ such that the sum $g_\lambda \eta_\lambda(x_3) + \ell_\rho F_\rho(\xi, \eta)$ vanishes for every set of admissible variations, and equations (2.9) and (2.10) with $\ell_0 = 0$ can be proved to hold in the same way as before.

THEOREM 1. *If an admissible arc E_{32} that satisfies the multiplier rule is normal, then ℓ_0 is different from zero and may be taken equal to unity. The set $\ell_0 = 1, \ell_\rho, \lambda_\alpha$ for E_{32} is then unique.*

The equations (2.9) and (2.10) with reductions similar to those of (2.5) imply that

$$\begin{aligned} \Omega_{y_\lambda'}(x_3)\,\eta_\lambda(x_3) + \ell_\sigma F_\sigma &= \Omega_{y_\lambda'}(x_3)\,\eta_\lambda(x_3) - \Omega_{y_\lambda'}(x_2)\,\eta_\lambda(x_3) \\ &= -\int_{x_2}^{x_3} (\Omega_{y_\lambda}\eta_\lambda + \Omega_{y_\lambda'}\eta_\lambda')\,dx = 0 \end{aligned} \tag{4.2}$$

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[1] Cf. Bliss, Trans. XIX, p. 311; Lectures, p. 22. Bolza, Mathematische Annalen, vol. 74 (1913), p. 440.
[2] Bliss, Lectures, p. 23.

for all sets of admissible variations. Since the ℓ_σ are not all zero it follows from the normality hypothesis that ℓ_0 is not zero. The equations (2.9) are homogeneous in $\ell_\sigma, \lambda_\alpha(x_2)$ and (2.10) in $\lambda_\alpha(x), c_i$ so therefore ℓ_0 may be taken equal to 1. The set $\ell_0=1, \ell_\rho, \lambda_\alpha$ is then unique, for suppose there is another effective set L_σ, Λ_α with $L_0=1$. Then the set $\ell_\sigma - L_\sigma, \lambda_\alpha - \Lambda_\alpha$ satisfies (2.9) and (2.10) and has its first element zero. Hence from (4.2) and the definition of normality the $\ell_\sigma - L_\sigma$ are all zero. For the same reason the sums $[\lambda_\alpha(x_3) - \Lambda_\alpha(x_3)]\,\partial\varphi_\alpha(x_3)/\partial y_i'$ must all vanish. But these are the values of the c_i of (2.10) for the multipliers $\lambda_\alpha - \Lambda_\alpha$. Now the values $\lambda_\alpha - \Lambda_\alpha \equiv 0$ on x_3x_2 satisfy (2.10) for $c_i = 0$ and furthermore are the only such values, since by the argument following equations (2.4) the λ's satisfying (2.10) for given c's are unique. Thus $\lambda_\alpha \equiv \Lambda_\alpha$ on x_3x_2.

COROLLARY. *If an admissible arc E_{12} satisfies the multiplier rule and has a normal sub-arc E_{32} then E_{12} itself is normal.*

For if E_{12} is not normal it has a set of multipliers with $\ell_0=0$ which would be effective for E_{32}.

THEOREM 2. *If a p-parameter family of admissible arcs*

$$y_i = y_i(x,a) \quad [X_1 \leqq x_1(a) \leqq x \leqq x_2(a) \leqq X_2;\ |a_\pi - \bar{a}_\pi| < \delta;\ (a) = (a_1, \cdots, a_p)] \tag{4.3}$$

contains a normal admissible arc E_{12} for $a_\pi = \bar{a}_\pi$ and has the functions $y_i, y_i', y_{ia}, y_{ia}'$ continuous in (x,a) except possibly for a finite number of values of x in X_1X_2 then there exists a number ϵ such that all arcs of the family for which $|a_\pi - \bar{a}_\pi| < \epsilon$ are also normal.

Let the equations (4.3) be substituted for the y's in equations (1.4). This gives

(4.4) $\eta_i \frac{\partial}{\partial y_i} \varphi_k(x, y(x,a), y'(x,a)) + \eta_i' \frac{\partial}{\partial y_i'} \varphi_k(x, y(x,a), y'(x,a)) - \xi_k = 0$

as the equations defining admissible variations for the curves of the family (4.3). Suppose a set of admissible variations $\bar{\eta}_i(x)$ is known for E_{12}. The equations (4.4) with $a_\pi = \bar{a}_\pi$ then determine corresponding admissible variations $\bar{\xi}_i$. The coefficients of η_i, η_i' in (4.4) are continuous in (x,a) except possibly for a finite number of values of x, and the functional determinant with respect to η_i' is $|\partial \varphi_k / \partial y_i'|$, which is different from zero along E_{12}. Hence, letting the ζ_i remain equal to $\bar{\xi}_i$, the imbedding theorem[1] for differential equations implies that for every set $\eta_i(x), a_\pi$ near $\bar{\eta}_i(x), \bar{a}_\pi$ there exist solutions

(4.5) $$\eta_i = \eta_i(x, \eta_1, a)$$

of equations (4.4) of class D′ in their arguments.

Now since E_{12} is normal there exist for it $n+r$ sets of admissible variations $\bar{\xi}_i^\omega, \bar{\eta}_i^\omega$ such that the determinant $|\bar{\eta}_{i1}^\omega \; F_\rho(\bar{\xi}^\omega, \bar{\eta}^\omega)|$ is different from zero. This determinant is continuous in its arguments $x_2, y_{i2}, \xi_2^\omega, \eta_{i1}^\omega, \eta_{i2}^\omega$. By the results of the preceding paragraph, for every set of initial values η_{i1}^ω, a_π in a neighborhood of $\bar{\eta}_{i1}^\omega, \bar{a}_\pi$ there exist variations

$$\eta_i^\omega = \eta_i^\omega(x, \eta_1^\omega, a)$$

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[1] Bolza, loc. cit., p. 179, with modifications to include parameters (cf. ibid., p. 187) and to apply to arcs E_{12} of class D′.

admissible for the arc given by the parameter values a_π and having the continuity properties of (4.5). Now if these values are substituted for $\bar{\eta}_i^\omega$, and $x_2(a)$, $y_i(x_2(a),a)$, ξ_2^ω for x_2, y_{i2}, $\bar{\xi}_2^\omega$, respectively, then the determinant mentioned above becomes a continuous function of ξ_2^ω, η_{i1}^ω, a_π and hence remains different from zero in a neighborhood of $\bar{\xi}_2^\omega$, $\bar{\eta}_{i1}^\omega$, $\bar{a}_\pi$. That is, every arc of the family (4.3) for which a_π is near $\bar{a}_\pi$ has a system of admissible variations with which it satisfies the definition of normality.

5. ANALOGUES OF THE WEIERSTRASS AND LEGENDRE CONDITIONS

The methods of this section are extensions of those used by Bliss[1] for the Lagrange problem with fixed end points and by Miss G. A. Larew[2] for the classical Mayer problem.

If the sub-arc E_{32} of E_{12} is normal, there exist n+r sets of admissible variations $\xi_2^\omega, \eta_i^\omega$ such that the determinant

$$\left| \eta_i^\omega(x_3) \quad F_\rho(\xi^\omega, \eta^\omega) \right| \tag{5.1}$$

is not zero. By Lemma 2 of Section 1 there is an (n+r)-parameter family of admissible arcs

$$y_i = y_i(x, b) \qquad [(b) = (b_1, \cdots, b_{m+r})] \tag{5.2}$$

containing E_{32} for $b_\omega = 0$ and having η_i^ω as variations along E_{32}.

A set of values (x, y, y') will be called admissible if it lies in the region R of page 1, satisfies the equations $\varphi_\alpha = 0$, and gives the matrix $\| \partial \varphi_\alpha / \partial y_i' \|$ the rank m. Let Y_{i3}' denote any value such that (x_3, y_3, Y_3') is admissible. Through this set there exists an admissible arc

$$y_i = Y_i(x) \qquad (x_3 \leqq x < x_3 + \epsilon)$$

of class C'. The equations

$$y_i(x_4, b) = Y_i(x_4) \qquad (x_3 \leqq x_4 < x_3 + \epsilon),$$

$$f_\rho(x_2(b), y(x_2(b), b)) = 0,$$

where

$$x_2(b) = \bar{x}_2 + b_\omega \xi_2^\omega,$$

have the initial solution $(x_4, b) = (x_3, 0)$ at which their functional determinant with respect to the b's is (5.1). Hence they

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[1] Lectures, p. 49.
[2] Trans. XX, p. 10.

determine functions $b_\omega = B_\omega(x_4)$ that vanish for $x = x_4$ and are of class C' in x_4 for some neighborhood $x_3 \leqq x_4 < x_3 + \delta$ provided that the right hand limiting values of $y_i'(x, b)$ at $x = x_3$ are taken if 3 is a corner of E_{12}. The equations (5.2) now give a one-parameter family

$$y_i = y_i(x, B(x_4)) \equiv \mathcal{Y}_i(x, x_4),$$
$$x_2 = \bar{x}_2 + B_\omega(x_4)\xi_2^\omega \equiv \mathcal{X}_2(x_4),$$

having the properties

$$\mathcal{Y}_i(x, x_3) = y_i(x), \quad \mathcal{Y}_i(x_4, x_4) = Y_i(x_4),$$
$$f_\rho(\mathcal{X}_2(x_4), \mathcal{Y}(\mathcal{X}_2(x_4), x_4)) = 0, \tag{5.3}$$
$$\varphi_\alpha(x, \mathcal{Y}(x, x_4), \mathcal{Y}'(x, x_4)) = 0$$

for $x_4 \leqq x \leqq \mathcal{X}_2(x_4)$, $x_3 \leqq x_4 < x_3 + \delta$. Denote the functions $\partial \mathcal{Y}_i(x, x_3)/\partial x_4$ by $\eta_i(x)$. They are admissible variations for E_{12} on $x_3 x_2$, and hence

$$0 = \int_{x_3}^{x} (\Omega_{y_i}\eta_i + \Omega_{y_i'}\eta_i')\,dx = \Omega_{y_i'}(x)\eta_i(x) - \Omega_{y_i'}(x_3)\eta_i(x_3). \tag{5.4}$$

Therefore $\Omega_{y_i'}(x)\eta_i(x)$ is constant on $x_3 x_2$.

From the equations of the first line of (5.3) it follows that

$$\frac{\partial}{\partial x_4}\mathcal{Y}_i(x_3, x_3) + \mathcal{Y}_i'(x_3, x_3) = \eta_i(x_3) + y_i'(x_3) = Y_i'(x_3). \tag{5.7}$$

From the second line of (5.4) we have

$$\left(\frac{\partial f_\rho}{\partial x_2} + \frac{\partial f_\rho}{\partial y_{i2}} y_{i2}'\right)\xi_2 + \frac{\partial f_\rho}{\partial y_{i2}}\eta_{i2} = 0, \tag{5.8}$$

where $\xi_2 = \frac{\partial}{\partial x_4}\mathcal{X}_2(x_4)$ and the arguments of the f's are those of E_{12}.

Now if E_{12} is a minimizing arc it must be true that

$$\left[\frac{d}{dx_4} f_\rho(\mathcal{X}_2(x_4), \mathcal{Y}(\mathcal{X}_2(x_4), x_4))\right]_{x_4 = x_3} = \left(\frac{\partial f_\rho}{\partial x_2} + \frac{\partial f_\rho}{\partial y_{i2}} y_{i2}'\right)\xi_2 + \frac{\partial f_\rho}{\partial y_{i2}}\eta_{i2} \geqq 0. \tag{5.9}$$

The transversality conditions

$$\ell_\sigma\Big(\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}}\, y_{i2}'\Big) = 0,$$

$$\ell_\sigma \frac{\partial f_\sigma}{\partial y_{i2}} + \Omega_{y_i'}(x_2) = 0,$$

with $\ell_0 = 1$, must also hold for E_{12}. On multiplying these equations by ξ_2, η_{i2} respectively and adding, we have by virtue of (5.8) and (5.9)

$$\Big[\frac{df_0}{dx_4}\Big]_{x_4 = x_3} = -\Omega_{y_i'}(x_2)\,\eta_{i2} \geqq 0. \tag{5.10}$$

Since $\Omega_{y_i'}\eta_i$ has been shown to be constant on x_3x_2, x_2 may be replaced by x_3 in the above equation. The use of (5.7) now gives the analogue of the Weierstrass condition:

THEOREM 1. *If E_{12} is a minimizing arc, then at every point 3 on E_{12} for which E_{32} is normal the inequality*

$$(y_{i3}' - Y_{i3}')\,\Omega_{y_i'}(x_3, y_3, y_3', \lambda_3) \geqq 0 \tag{5.11}$$

must hold for every set Y_{i3} such that (x_3, y_3, Y_3') is admissible.

The inequality (5.11) holds at corners of E_{12} for left hand as well as for right hand limiting values of y_i', λ_α since the left member is continuous in all its arguments.

The left member of (5.11) bears closer resemblance to the Weierstrass E-function if the vanishing expression $\Omega(x_3, y_3, Y_3', \lambda_3) - \Omega(x_3, y_3, y_3', \lambda_3)$ is added.

To derive an analogue of the Legendre condition consider a set of values π_i satisfying the equations

$$\pi_i \frac{\partial \varphi_\alpha}{\partial y_i'} = 0 \tag{5.12}$$

at an element (x_3, y_3, y_3') of E_{12}. By means of the functions φ_τ

adjoined to the set φ_α in Section 1, the equations

$$\pi_i \frac{\partial}{\partial y_i'} \varphi_\tau(x_3, y_3, y_3') = \vartheta_\tau \tag{5.13}$$

define n-m further quantities ϑ_τ. The equations

$$\varphi_\alpha(x_3, y_3, p) = 0, \quad \varphi_\tau(x_3, y_3, p) = z_\tau + \epsilon \vartheta_\tau$$

have the initial solution $(\epsilon, p) = (0, y_3')$ and determine uniquely a set of solutions $p_i(\epsilon)$ with initial values $p_i(0) = y_{i3}'$. The derivatives $p_{i\epsilon}(0)$ satisfy equations (5.12) and (5.13) when substituted for the π_i and hence coincide with them. The sets $(x_3, y_3, p(\epsilon))$ are all admissible, and hence by Theorem 1

$$(y_{i3}' - p_i(\epsilon))\, \Omega_{y_i'}(x_3, y_3, y_3', \lambda_3) \geqq 0. \tag{5.14}$$

The equality holds when $\epsilon = 0$. Also the first derivative with respect to ϵ of the left member vanishes when $\epsilon = 0$, since

$$\Omega(x_3, y_3, p(\epsilon), \lambda_3) = 0 \tag{5.15}$$

identically in ϵ, and the derivative of this function with $\epsilon = 0$ is the negative of that of (5.14). The second derivative of (5.14) must then be greater than or equal to zero for $\epsilon = 0$, and this derivative is equal to

$$p_{i\epsilon}(0)\, p_{k\epsilon}(0)\, \Omega_{y_i' y_k'}(x_3, y_3, y_3', \lambda_3),$$

as is evident from differentiating (5.15) twice.

This gives as an analogue to Legendre's condition:

THEOREM 2. <u>If E_{12} is a minimizing arc, then at every point 3 on E_{12} for which E_{32} is normal the inequality</u>

(26)

$$\Omega_{x'_i y'_k}(x_3, y_3, y'_3, \lambda_3)\,\pi_i \pi_k \geqq 0$$

must be satisfied by every set π_i which is a solution of the m equations

$$\pi_i \frac{\partial}{\partial y'_i} \varphi_\alpha(x_3, y_3, y'_3) = 0.$$

6. AN ENVELOPE THEOREM

An envelope theorem will be proved for a one-parameter family of extremals having an envelope, irrespective of how such a family might have been obtained. From this envelope theorem an analogue of Jacobi's condition will be derived. Later, methods will be given for selecting from the 2n-parameter family of extremals a one-parameter family having an envelope.

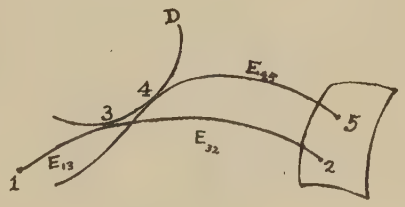
Consider a one-parameter family of extremals

(6.1) $\qquad y_i = Y_i(x,t), \quad \lambda_\alpha = \Lambda_\alpha(x,t)$

which contains the normal extremal E_{12} for $t=\bar{t}$, satisfies the end point relations $f_\rho = 0$ and the transversality conditions (2.9) for $x = x_2(t)$, and has an envelope D with equations

(6.2) $\qquad x = X(t), \quad y_i = Y_i(X(t),t).$

Suppose the envelope has a contact point 3 with E_{12} between 1 and 2, the coordinates of which are given by (6.2) with $t=\bar{t}$, and assume further that $X_t(\bar{t}) \neq 0$, which insures that a branch of the envelope projects from 3 towards 2. Let the functions Y_i be of class C" and X and Λ_α of class C' in their arguments. Denote by $E_{13} + D_{34} + E_{45}$ the composite arc made up of the part of E_{12} between 1 and 3, the envelope D from 3 to a neighboring point 4 on D, and the arc E_{45} of the extremal which touches D at 4 and satisfies the equations $f_\rho = 0$ at 5.



THEOREM. For 4 near 3 the composite arcs $E_{13}+D_{34}+E_{45}$ of a one-parameter family of extremals with an envelope such as described are all admissible and give f_0 the same value.

That the composite arcs are admissible is evident from the fact that the envelope and an extremal of the family have the same direction at their point of contact. From this fact it also follows that

$$\begin{aligned} &\rho(t)\cdot 1 \equiv \bar{X}_t(t), \\ (6.3)\qquad &\rho(t)\cdot Y_i'(\bar{X}(t),t) \equiv Y_i'(\bar{X}(t),t)\,\bar{X}_t(t) + Y_{it}(\bar{X}(t),t), \end{aligned}$$

where $\rho(t)$ is a proportionality factor. The symbol $\equiv$ is used here and in the remainder of the proof to denote identities in t, except where otherwise specified. If the first equation of (6.3) is used to eliminate $\rho(t)$, the remaining reduce to

$$Y_{it}(\bar{X}(t),t) \equiv 0. \tag{6.4}$$

The second end points of the family (6.1) are given by

$$x_2 = x_2(t), \quad y_{i2} = Y_i(x_2(t),t). \tag{6.5}$$

With these arguments in f_0, it follows that

$$\frac{df_0}{dt} = \frac{\partial f_0}{\partial x_2}x_{2t} + \frac{\partial f_0}{\partial y_{i2}}\left(Y_i'x_{2t} + Y_{it}\right)\Big|^{x=x_2(t)} \tag{6.6}$$

The end relations $f_\rho = 0$ are satisfied by the values (6.5) and hence

$$\frac{\partial f_\rho}{\partial x_2}x_{2t} + \frac{\partial f_\rho}{\partial y_{i2}}\left(Y_i'x_{2t} + Y_{it}\right)\Big|^{x=x_2(t)} \equiv 0. \tag{6.7}$$

From the transversality conditions (2.9) the equations

$$\begin{aligned} &\ell_\sigma\left(\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}}Y_i'\right)\Big|^{x=x_2(t)} \equiv 0, \\ (6.8)\qquad &\ell_\sigma\frac{\partial f_\sigma}{\partial y_{i2}} + \Omega_{y_i'}\Big|^{x=x_2(t)} \equiv 0 \end{aligned}$$

also hold, where the ℓ's are functions of t and Ω has the arguments (x, Y, Y', Λ). Multiply the above equations by x_{2t}, Y_{it}

respectively, and add. This gives

$$(6.9)\qquad \ell_\sigma\Big[\frac{\partial f_\sigma}{\partial x_2}x_{2t}+\frac{\partial f_\sigma}{\partial y_{i2}}(Y_i'x_{2t}+Y_{it})\Big]+\Omega_{y_i'}Y_{it}\Big|^{x=x_1(t)}\equiv 0,$$

which, by (6.6) and (6.7), reduces to

$$(6.10)\qquad \ell_0\frac{df_0}{dt}\equiv-\Omega_{y_i'}Y_{it}\Big|^{x=x_1(t)}.$$

The equations

$$\Omega(x,Y(x,t),Y'(x,t),\lambda)\equiv 0$$

hold identically in x,t,λ_α. Differentiation as to t gives

$$\Omega_{y_i}Y_{it}+\Omega_{y_i'}Y_{it}'\equiv(\Omega_{y_i}-\frac{d}{dx}\Omega_{y_i'})Y_{it}+\frac{d}{dx}\Omega_{y_i'}Y_{it}\equiv 0,$$

from which it is evident that for $\lambda_\alpha=\Lambda_\alpha(x,t)$ the sum $\Omega_{y_i'}Y_{it}$ is independent of x. It is furthermore identically zero in t, since by (6.4) it vanishes for $x=X(t)$. Therefore from (6.10)

$$\ell_0\frac{df_0}{dt}\equiv 0,$$

identically in t. Since E_{12} is assumed normal, the arcs of the family (6.1) are also normal for t near $\bar t$ by Theorem 2 of Section 4. By Theorem 1 of the same section it is then true that $\ell_0\neq 0$ for t near $\bar t$. Hence f_0 is constant in this neighborhood of $\bar t$. This completes the proof of the theorem.

7. AN ANALOGUE OF JACOBI'S CONDITION

Consider again the one-parameter family of extremals (6.1) having the envelope (6.2). A contact point of an extremal of the family with the envelope will be called a <u>focal point</u> on the extremal of the hypersurface defined by the second end point equations $f_\rho = 0$. An analogue to Jacobi's condition can now be stated as follows:

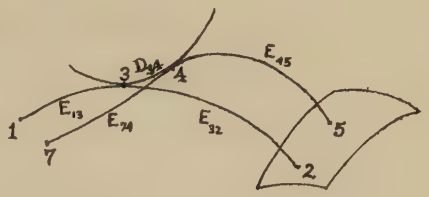
THEOREM. <u>If an extremal E_{12} minimizes f_0 and has the determinant R different from zero along it, then between 1 and 2 or at 1 there can be no focal point 3 of such a nature that E_{32} is normal, $X_\xi(\bar{t})$ is not zero, and $Y_{i\xi}(X(\bar{t}),\bar{t})$ are not all zero.</u>

The property that $Y_{i\xi}(X(\bar{t}),\bar{t})$ are not all zero will be shown in Section 8 to be a consequence of the assumption $\Delta_\chi(\chi_3) \neq 0$ which is needed there in obtaining a one-parameter family of extremals having an envelope.

For the proof it will be assumed that a minimizing arc E_{12} along which R is not zero does have a focal point 3 between 1 and 2 or at 1 for which the three final properties mentioned in the theorem are true. The theorem of Section 6 then requires every composite arc $E_{13} + D_{34} + E_{45}$ for 4 near 3 to be a minimizing arc. This will be shown to imply that the composite arc $E_{13} + D_{34} + E_{45}$ is an extremal having the same values $x, y_i, y_i', \lambda_\alpha$ at the point 4 as has the extremal E_{745} of which E_{45} is a part. The uniqueness property of extremals then requires E_{745} to coincide with the envelope from 3 to 4. Finally, this will be proved to lead to the contradiction that the $Y_{i\xi}'(X(\bar{t}),\bar{t})$ are all zero.

Denote by $T(x)$ the solution of the equation $x = X(t)$ for t. Let the equations of the composite arc $E_{13} + D_{34} + E_{45}$ be

(7.1) $\quad y_i = \mathcal{Y}_i(x,t)$



where

$$\mathcal{Y}_i(x,t) = Y_i(x,\bar{t}) \quad \text{on} \quad x_1 \leqq x \leqq X(\bar{t}) = x_3,$$
$$= Y_i(x,T(x)) \quad \text{on} \quad X(\bar{t}) \leqq x \leqq X(t) = x_4,$$
$$= Y_i(x,t) \quad \text{on} \quad X(t) \leqq x \leqq x_2(t) = x_5.$$

The functions $\mathcal{Y}_i, \mathcal{Y}_i'$ are continuous in (x,t) for $x_1 \leqq x \leqq x_5$ and t near $\bar{t}$. The same is true of $\mathcal{Y}_{it}$ and $\mathcal{Y}_{it}'$ except for $x = x_4$. Hence, since E_{32} is normal, the corollary and Theorem 2 of Section 4 imply that for t near $\bar{t}$ the arcs $E_{13}+D_{34}+E_{45}$, $D_{34}+E_{45}$, E_{45}, and E_{745} are normal. From Theorem 1 of Section 4 and the multiplier rule it now follows that these arcs each have unique sets of multipliers $\ell_0 = 1, \ell_\rho, \lambda_\alpha$, and also that the λ's are continuous since the arcs are all without corners. Denote the λ-multipliers belonging to $E_{13} + D_{34} + E_{45}$ by $\mathcal{L}_\alpha(x,t)$. On x_4x_5 these multipliers with the corresponding $\ell_0 = 1, \ell_\rho$ are effective for E_{45} and hence coincide with the unique set for that arc. Similarly, the unique multipliers of E_{745} coincide with those of E_{45}, so that on x_4x_5 $\mathcal{L}_\alpha(x,t)$, $\Lambda_\alpha(x,t)$ are equal, provided the latter functions for the family (6.1) are so chosen that the corresponding ℓ_0's are equal to 1. Thus it is established that the values $x, y_i, y_i', \lambda_\alpha$ of $E_{13} + D_{34} + E_{45}$ and E_{745} are the same at 4.

To show that $E_{13} + D_{34} + E_{45}$ is an extremal we have yet to establish the continuity properties required in the definition. To this effect we may note that the $\mathcal{L}$'s are continuous in x and

t and that the determinant R is continuous in $x, y_i, y_i', \lambda_\alpha$. Hence R is a continuous function of x and t for $y_i = \eta_i, \lambda_\alpha = \Lambda_\alpha$ and is different from zero along this arc if t is near $\bar{t}$, since it is different from zero on E_{12}. By Corollary 3, Section 2, the composite arc then has the desired continuity properties.

Now by the differential equation theory of Section 3 there is a unique set of functions y_i, λ_α satisfying the Euler-Lagrange differential equations (2.11), having the continuity properties specified in the definition of extremals, and taking on a given set of initial values x_0, y_{i0}, v_{i0}, or what amounts to the same thing, a set of initial values of $x, y_i, y_i', \lambda_\alpha$. Hence the arcs $E_{13} + D_{34} + E_{45}$ and E_{745} coincide on $x_1 x_5$.

If the extremal E_{745} corresponds to the parameter value $\bar{t} + h$ then all the extremals (6.1) for $\bar{t} \leqq t \leqq \bar{t} + h$ coincide with the envelope D from the points $[X(t), Y_i(X(t), t)]$ to the point 3. This implies the identities in t and t°

$$Y_i(X(t), t^\circ) \equiv Y_i(X(t), t) \qquad (\bar{t} \leqq t \leqq t^\circ \leqq \bar{t} + h).$$

Differentiation as to t° gives

$$Y_{it}(X(t), t^\circ) \equiv 0,$$

and of this as to t,

$$Y'_{it}(X(t), t^\circ) X_t(t) \equiv 0.$$

Since $X_t(\bar{t}) \neq 0$ we obtain from the latter, on putting $t = t^\circ = \bar{t}$,

$$Y'_{it}(X(\bar{t}), \bar{t}) = 0.$$

This gives the desired contradiction and completes the proof of the theorem.

8. DETERMINATION OF A ONE-PARAMETER FAMILY OF EXTREMALS HAVING AN ENVELOPE

In this section a method is given for obtaining directly from the 2n-parameter family of extremals a one-parameter family having an envelope of the kind needed in Sections 6 and 7.[1] The focal points are shown to be the zeros of a certain determinant. Following that a second method of determining a one-parameter family of extremals having an envelope is considered, and it is proved that the corresponding determinant gives the same focal points as are obtained by the first method.

Let E_{12} be a normal extremal which satisfies the transversality conditions and along which the determinant R is not zero, and let the equations of the 2n-parameter family of extremals containing E_{12} be taken in the form

$$(8.1) \qquad y_i = y_i(x, x_2, y_2, v_2), \quad \lambda_\alpha = \lambda_\alpha(x, x_2, y_2, v_2),$$

where only 2n of the parameters x_2, y_{i2}, v_{i2} are independent, as was shown in Section 34. Since E_{12} is normal, the curves of the family (8.1) in a neighborhood of E_{12} are also normal by Theorem 2, Section 4, and hence ℓ_0 may be taken equal to 1 in this neighborhood.

Now it is desired to obtain functions $X(t), x_2(t), y_{i2}(t), v_{i2}(t), \ell_\rho(t)$ such that the one-parameter family

$$(8.2) \qquad y_i = y_i(x, x_2(t), y_2(t), v_2(t)) \equiv Y_i(x, t)$$

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[1] For the corresponding theory for Lagrange problems with fixed end points see Bolza, Vorlesungen über Variationsrechnung, p. 613; Bliss, Lectures, p. 51. Cf. also Mackie, Annals of Mathematics, loc. cit., p. 399.

shall contain E_{12} and have an envelope D

$$(8.3)\qquad x = X(t),\ y_i = y_i\big(X(t), x_2(t), y_2(t), v_2(t)\big) \equiv Y_i\big(X(t), t\big)$$

and shall satisfy identically in t the equations

$$(8.4)\qquad \begin{aligned} &f_\rho(x_2, y_2) = 0,\\ &\frac{\partial}{\partial x_2} f(x_2, y_2, \ell) - v_{i2}\, y_i'(x_2, x_2, y_2, v_2) = 0,\\ &\frac{\partial f}{\partial y_{i2}} + v_{i2} = 0, \end{aligned}$$

for t near $\bar{t}$, where f denotes the function $f_0 + \ell_\rho f_\rho$. The second and third lines of (8.4) are the forms to which the transversality conditions reduce when the terms $\ell_\sigma y_{i2}', \partial f_\sigma / \partial y_{i2}$ in the first equation of (2.9) are eliminated by use of the remaining equations and when the $\Omega_{y_i'}(x_2)$ are replaced by v_{i2}, to which they are equal by (3.1).

The envelope and a curve of the family must have the same direction at their point of contact. Hence the first of the following equations can be obtained in the same way as were equations (6.4), and the remaining can be found by differentiating (8.4)[1]:

$$(8.5)\qquad \begin{aligned} &\frac{\partial y_i}{\partial x_2} x_{2t} + \frac{\partial y_i}{\partial y_{\kappa 2}} y_{\kappa 2 t} + \frac{\partial y_i}{\partial v_{\kappa 2}} v_{\kappa 2 t} = 0,\\ &\frac{\partial f_\rho}{\partial x_2} x_{2t} + \frac{\partial f_\rho}{\partial y_{\kappa 2}} y_{\kappa 2 t} = 0,\\ &\left[\frac{\partial^2 f}{\partial x_2^2} - v_{i2}\left(y_i''(x_2) + \frac{\partial}{\partial x_2} y_i'(x_2)\right)\right] x_{2t} + \left[\frac{\partial^2 f}{\partial x_2 \partial y_{\kappa 2}} - v_{i2} \frac{\partial}{\partial y_{\kappa 2}} y_i'(x_2)\right] y_{\kappa 2 t}\\ &\qquad - \left[y_\kappa'(x_2) + v_{i2} \frac{\partial}{\partial v_{\kappa 2}} y_i'(x_2)\right] v_{\kappa 2 t} + \frac{\partial f_\rho}{\partial x_2} \ell_{\rho t} = 0,\\ &\frac{\partial^2 f}{\partial y_{i2} \partial x_2} x_{2t} + \frac{\partial^2 f}{\partial y_{i2} \partial y_{\kappa 2}} y_{\kappa 2 t} + v_{i2 t} + \frac{\partial f_\rho}{\partial y_{i2}} \ell_{\rho t} = 0. \end{aligned}$$

---

[1] Since y_{i2} is being used as a parameter we shall represent $\big[y_i'(x, x_2, y_2, v_2)\big]_{x = x_2}$ by $y_i'(x_2)$ instead of y_{i2}'.

In order to show that the terms $v_{i_2}\,\partial y_i'(x_2)/\partial v_{k_2}$ are identically zero, we note first that the functions $\varphi_\alpha(x, y, \psi(x, y, v))$ vanish identically, by (3.1) and (3.2), and hence the sums $\pi_\alpha\,\partial\varphi_\alpha/\partial y_i'\cdot\partial\psi_i/\partial v_k$ also vanish. But (3.1) and (3.2) imply that these sums are equal to the sums $v_\lambda\,\partial\psi_i/\partial v_k$, and from this the conclusion follows, since

$$y_i'(x_2, x_2, y_2, v_2) \equiv \psi_i(x_2, y_2, v_2).$$

The desired ends will now be attained if functions $X(t), x_2(t), y_{i_2}(t), v_{i_2}(t), \ell_\rho(t)$ can be found satisfying the above differential equations (8.5) and the initial conditions

(8.6) $\quad x_2(\bar t) = \bar x_2\,,\ y_{i_2}(\bar t) = \bar y_{i_2}\,,\ v_{i_2}(\bar t) = \bar v_{i_2}\,,\ \ell_\rho(\bar t) = \bar\ell_\rho\,.$

The equations (8.5) will have solutions $x_{2t}, y_{k_2t}, v_{k_2t}, \ell_{\rho t}$ in terms of $x, x_2, y_{i_2}, v_{i_2}, \ell_\rho$ and not all zero if and only if the determinant

$$\Delta(x, x_2, y_2, v_2, \ell) = \begin{vmatrix} \dfrac{\partial y_i}{\partial x_2} & \dfrac{\partial y_i}{\partial y_{k_2}} & \dfrac{\partial y_i}{\partial v_{k_2}} & 0 \\ \dfrac{\partial f_\rho}{\partial x_2} & \dfrac{\partial f_\rho}{\partial y_{k_2}} & 0 & 0 \\ \dfrac{\partial^2 f}{\partial x_2^2} - v_{i_2}\left(y_i''(x_2) - \dfrac{\partial}{\partial x_2}y_i'(x_2)\right) & \dfrac{\partial^2 f}{\partial x_2\partial y_{k_2}} - v_{i_2}\dfrac{\partial}{\partial y_{k_2}}y_i'(x_2) & -y_k'(x_2) & \dfrac{\partial f_\rho}{\partial x_2} \\ \dfrac{\partial^2 f}{\partial y_{i_2}\partial x_2} & \dfrac{\partial^2 f}{\partial y_{i_2}\partial y_{k_2}} & \delta_{ik} & \dfrac{\partial f_\rho}{\partial y_{i_2}} \end{vmatrix}$$

vanishes, where δ_{ik} is the Kronecker symbol. Hence let x be determined as a function of $x_2, y_{i_2}, v_{i_2}, \ell_\rho$ by setting $\Delta = 0$. When $x = x_2$ the elements $\partial y_i/\partial v_{k_2}$ all vanish, and hence Δ also vanishes, unless $r = 1$. If $r = 1$, Δ does not vanish, provided the functions $\partial f_1/\partial x_2$ and $\partial f_1/\partial x_2 + y_i'(x_2)\,\partial f_1/\partial y_{i_2}$ are not zero. Assume then that $\Delta(x, \bar x_2, \bar y_2, \bar v_2, \bar\ell) = 0$ has another solution

$x = x_3, \bar{x}_1 \leq x_3 < \bar{x}_2$ and that

$$\Delta_x(x_3, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell}) \neq 0. \tag{8.7}$$

Then, for (x_2, y_2, v_2, ℓ) in a neighborhood of $(\bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$ the equation $\Delta(x, x_2, y_2, v_2, \ell) = 0$ will have a solution

$$x = x(x_2, y_2, v_2, \ell) \tag{8.8}$$

with the initial value $x_3 = x(\bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$. The rank of $\Delta(x_3, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$ is 2n+r, as a consequence of (8.7). Hence, if the cofactor of the element in the μ^{th} row and ν^{th} column of Δ is denoted by $\Delta_{\mu\nu}$, then at least one $\Delta_{\mu\nu}(x_3, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$ is not zero, say for $\mu = a, \nu = b$. Therefore, when (8.8) is substituted in (8.5) the latter will have solutions in the form

$$\begin{aligned} x_{2t} &= c\,\Delta_{a,1}, \\ y_{k2t} &= c\,\Delta_{a,1+k}, \\ v_{k2t} &= c\,\Delta_{a,m+1+k}, \\ \ell_{\rho t} &= c\,\Delta_{a,2m+1+\rho}, \end{aligned} \tag{8.9}$$

when the arguments (x_2, y_2, v_2, ℓ) of $\Delta_{a\nu}$ are near $(\bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$. Here c denotes a proportionality factor that may be taken equal to unity without loss of generality. Now the equations (8.9) have solutions

$$x_2 = x_2(t), \; y_{k2} = y_{k2}(t), \; v_{k2} = v_{k2}(t), \; \ell_\rho = \ell_\rho(t), \tag{8.10}$$

with initial values (8.6) and holding for t near $\bar{t}$. On substituting these solutions in (8.8), we have also a function

$$x = x(x_2(t), y_2(t), v_2(t), \ell(t)) \equiv X(t). \tag{8.11}$$

The values (8.10) when put in equations (8.1) give the desired one-parameter family of extremals and multipliers, and $x = X(t)$ then gives the envelope. To determine the class of these resulting functions, we note that, by Hypothesis 3 of

Section 1 and the Theorem of Section 3, Δ is of class C' in its arguments (x, x_2, y_2, v_2, ℓ), and hence the right members of (8.8) and (8.9) are of such class. The solutions (8.10) are therefore of class C'' and (8.11) of class C' in t. From the use of the Theorem of Section 3 again, it now follows that the functions defining the one-parameter family of extremals and multipliers are of class C'' in x and t and those for the envelope are of class C' in t.

The focal points on E_{12} are given by the zeros of $\Delta(x, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell})$ together with $y_i(x, \bar{x}_2, \bar{y}_2, \bar{v}_2)$, or by $X(\bar{t}), Y_i(X(\bar{t}), \bar{t})$.

To prove that $\Delta_x(x_1, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell}) \neq 0$ implies that $Y_{it}(X(\bar{t}), \bar{t})$ are not all zero, we note that only the first n rows of Δ are functions of x and hence Δ_x is equal to the sum of the n determinants obtained by substituting in turn a row of coefficients from

$$\frac{\partial y_i'}{\partial x_2} x_{2t} + \frac{\partial y_i'}{\partial y_{k2}} y_{k2t} + \frac{\partial y_i'}{\partial v_{k2}} v_{k2t} \tag{8.12}$$

for the corresponding row of elements in Δ. The expressions (8.12), however, are equal to $Y_{it}'(x,t)$, so that, if the latter vanish for $X(\bar{t}), \bar{t}$, then the former do also. The 3n+r+1 equations obtained by adjoining the expressions (8.12) set equal to zero to the equations (8.5) then have solutions x_{2t}, y_{i2t}, v_{i2t} $\ell_{\rho t}$, not all zero, for $t=\bar{t}$. Therefore the n determinants of which Δ_x is the sum must all vanish at $t=\bar{t}$, in contradiction of the assumption that $\Delta_x(x_1, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{\ell}) \neq 0$.

The results of this section may be summarized in the theorem:

THEOREM. Let E_{12} be a normal extremal that satisfies the transversality conditions, has the determinant R different from zero along it, and which is contained in the 2n-parameter family of extremals (8.1) for $(x_2, y_2, v_2) = (\bar{x}_2, \bar{y}_2, \bar{v}_2)$. Let $\Delta(x, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{l})$ have a zero $x = x_3$, $\bar{x}_1 \leqq x_3 < \bar{x}_2$, such that $\Delta_x(x_3, \bar{x}_2, \bar{y}_2, \bar{v}_2, \bar{l}) \neq 0$. Then there exists a one-parameter family of extremals

$$(8.13) \qquad y_i = Y_i(x,t), \quad \lambda_\alpha = \Lambda_\alpha(x,t) \quad (x_1 \leqq x \leqq x_2(t);\ t_1 \leqq t \leqq t_2)$$

containing E_{12} for $t = \bar{t}$ and having an envelope

$$x = X(t), \quad y_i = Y_i(X(t), t)$$

that touches E_{12} at $x_3 = X(\bar{t})$, $y_{i3} = Y_i(X(\bar{t}), \bar{t})$ and for which the $Y'_{it}(X(\bar{t}), \bar{t})$ are not all zero. For $x = x_2(t)$ the arcs of the family (8.13) satisfy the second end point and transversality conditions. The functions Y_i and Λ_α are of class C'' in x and t and X is of class C' in t.

The property $X'(\bar{t}) \neq 0$ remains unestablished, and hence it is added as a restrictive hypothesis in applying the theory of Sections 6 and 7 to an envelope as here obtained.

A determinant of simpler form whose zeros will give the focal points can be obtained, if, instead of determining a one-parameter family of extremals having an envelope directly from the 2n-parameter family, we first eliminate n of the parameters by using the second end point and transversality relations.

Since the matrix $\| \partial f_\rho / \partial x_2 \quad \partial f_\rho / \partial y_{i2} \|$ is of rank r for x_2, y_{i2} near $\bar{x}_2, \bar{y}_{i2}$ the equations

$$f_\rho(x_2, y_2) = 0$$

can be solved for x_2, y_{i2} in terms of n+1-r parameters

$u_1, \cdots, u_{m+1-r}$. Denote these functions by

$$x_2 = x_2(u_\beta), \quad y_{i2} = y_{i2}(u_\beta). \tag{8.14}$$

The functional matrix of the transversality conditions

$$\begin{aligned} &\frac{\partial f_0}{\partial x_2} + \lambda_\rho \frac{\partial f_\rho}{\partial x_2} - \nu_{i2} y_i'(x_2) = 0, \\ &\frac{\partial f_0}{\partial y_{i2}} + \lambda_\rho \frac{\partial f_\rho}{\partial y_{i2}} + \nu_{i2} = 0 \end{aligned} \tag{8.15}$$

with respect to $\lambda_\rho, \nu_{\kappa 2}$ is

$$\left\| \begin{matrix} \dfrac{\partial f_\rho}{\partial x_2} & -y_\kappa'(x_2) - \nu_{i2} \dfrac{\partial}{\partial \nu_{\kappa 2}} y_i'(x_2) \\ \dfrac{\partial f_\rho}{\partial y_{i2}} & \delta_{i\kappa} \end{matrix} \right\|.$$

The $\nu_{i2} \partial y_i'(x_2)/\partial \nu_{\kappa 2}$ vanish identically, as explained under equations (8.5). Hence by a simple reduction we have

$$\left\| \begin{matrix} \dfrac{\partial f_\rho}{\partial x_2} + \dfrac{\partial f_\rho}{\partial y_{i2}} y_i'(x_2) & 0 \\ \dfrac{\partial f_\rho}{\partial y_{i2}} & \delta_{i\kappa} \end{matrix} \right\|. \tag{8.16}$$

Let it be assumed that $\partial f_\rho/\partial x_2 + y_i'(x_2)\, \partial f_\rho/\partial y_{i2}$ are not all zero. Then the matrix (8.16) is of rank r+1 and the equations (8.15), with $x_2(u_\beta), y_{i2}(u_\beta)$ substituted for x_2, y_{i2}, can be solved for ν_{i2}, λ_ρ in terms of $u_1, \cdots, u_{m+1-r}$ and r-1 other parameters $u_{m+2-r}, \cdots, u_m$. Denote these functions by

$$\lambda_\rho = \lambda_\rho(u_\kappa), \quad \nu_{i2} = \nu_{i2}(u_\kappa). \tag{8.17}$$

Thus is obtained an n-parameter family of extremals

$$\begin{aligned} y_i &= y_i(x, x_2(u_\beta), y_2(u_\beta), \nu_2(u_\kappa)) \equiv \mathfrak{y}_i(x, u_\kappa), \\ \lambda_\alpha &= \lambda_\alpha(x, x_2(u_\beta), y_2(u_\beta), \nu_2(u_\kappa)) \equiv \mathfrak{L}_\alpha(x, u_\kappa) \end{aligned} \tag{8.18}$$

which satisfies the second end point relations and transversality conditions for $x = x_2(u_\beta)$ and contains E_{12} for some set of

parameter values, say $\bar{u}_\kappa$.

The family (8.18) will contain a one-parameter family[1]

(8.19) $$y_i = \mathcal{Y}_i(x, u_\kappa(t)) \equiv Y_i(x,t), \quad \lambda_\alpha = \mathcal{L}_\alpha(x, u_\kappa(t)) \equiv \Lambda_\alpha(x,t)$$

having an envelope

(8.20) $$x = X(t), \quad y_i = Y_i(X(t),t),$$

if functions $X(t), u_\kappa(t)$ can be so chosen as to satisfy identically in t the equations

$$\frac{\partial \mathcal{Y}_i}{\partial u_\kappa} u_{\kappa t} = 0$$

To determine $X(t), u_\kappa(t)$ we proceed as before by first determining x as a function of u_κ by setting the determinant

(8.21) $$D = \left| \frac{\partial \mathcal{Y}_i}{\partial u_\kappa} \right| = \left| \frac{\partial y_i}{\partial x_2}\frac{\partial x_2}{\partial u_g} + \frac{\partial y_i}{\partial y_{\kappa 2}}\frac{\partial y_{\kappa 2}}{\partial u_g} + \frac{\partial y_i}{\partial v_{\kappa 2}}\frac{\partial v_{\kappa 2}}{\partial u_g} \quad \frac{\partial y_i}{\partial v_{\kappa 2}}\frac{\partial v_{\kappa 2}}{\partial u_k} \right|$$

equal to zero. As before, if $r > 1$, $x = x_2$ is a zero of the determinant. Assume that $x = x_3$, $x_1 \leqq x_3 < \bar{x}_2$, is a solution of $D(x, \bar{u}_\kappa) = 0$ and that $D_x(x_3, \bar{u}_\kappa) \neq 0$. Then $D(x, u_\kappa) = 0$ has a solution $x = x(u_\kappa)$ having the initial value $x_3 = x(\bar{u}_\kappa)$ and holding for u_κ near $\bar{u}_\kappa$. From differential equations similar to (8.9) solutions $u_\kappa(t)$ can now be obtained, which with $x(u_\kappa(t))$ give the one-parameter family (8.19) and envelope (8.20). The resulting theorem is the same as before if Δ is replaced by D and the hypothesis added that $\partial f_\rho / \partial x_2 + y_i'(x_2)\, \partial f_\rho / \partial y_{i2}$ are not all zero for $x = \bar{x}_2$, $y_{i2} = \bar{y}_{i2}$.

Geometrically, the vanishing of all the functions $\partial f_\rho / \partial x_2 + y_i'(x_2)\, \partial f_\rho / \partial y_{i2}$ for the arguments $\bar{x}_2, \bar{y}_{i2}$ would mean that the arc $y_i = \bar{y}_i(x)$ would be tangent to each of the $r+1$ hypersurfaces.

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[1] The notation Y_i, Λ_α is not intended to imply that the functions here are the same as those previously so denoted.

$$f_\rho(x_2, y_2) = 0$$

$$f_\sigma(x_2, y_2) = f_\sigma(\bar{x}_2, \bar{y}_2)$$

at the common point $\bar{x}_2, \bar{y}_{i2}$. For, the directions normal to the above hypersurfaces are proportional to $\partial f_\sigma / \partial x_2$, $\partial f_\sigma / \partial y_{i2}$, and those of the arc, to $1, y_i'(x_2)$. The orthogonality relation is then

$$\frac{\partial f_\sigma}{\partial x_2} + \frac{\partial f_\sigma}{\partial y_{i2}} y_i'(x_2) = 0.$$

It should be noted that if $\partial f_\rho / \partial x_2 + y_i'(x_2) \partial f_\rho / \partial y_{i2}$ all vanish then the first of the transversality conditions (2.9) implies that $\partial f_\sigma / \partial x_2 + y_i'(x_2) \partial f_\sigma / \partial y_{i2}$ also vanishes for normal arcs.

For values of their arguments belonging to E_{12} the roots of $\Delta(x, x_2, y_2, v_2, \ell) = 0$ and $D(x, u_\kappa) = 0$ are the same. To show this let Δ be written in the form

$$\Delta = \begin{vmatrix} \frac{\partial y_i}{\partial x_2} & \frac{\partial y_i}{\partial y_{\kappa 2}} & \frac{\partial y_i}{\partial v_{\kappa 2}} & 0 \\ \frac{\partial f_\rho}{\partial x_2} & \frac{\partial f_\rho}{\partial y_{\kappa 2}} & 0 & 0 \\ A & B_\kappa & 0 & \frac{\partial f_\rho}{\partial x_2} + \frac{\partial f_\rho}{\partial y_{i2}} y_i'(x_2) \\ \frac{\partial^2 f}{\partial y_{i2} \partial x_2} & \frac{\partial^2 f}{\partial y_{i2} \partial y_{\kappa 2}} & \delta_{i\kappa} & \frac{\partial f_\rho}{\partial y_{i2}} \end{vmatrix},$$

where

$$A = \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial y_{i2} \partial x_2} y_i'(x_2) - v_{i2} \left[y_i''(x_2) + \frac{\partial}{\partial x_2} y_i'(x_2) \right],$$

$$B_\kappa = \frac{\partial^2 f}{\partial x_2 \partial y_{\kappa 2}} + \frac{\partial^2 f}{\partial y_{i2} \partial y_{\kappa 2}} y_i'(x_2) - v_{i2} \frac{\partial}{\partial y_{\kappa 2}} y_i'(x_2).$$

Multiply Δ by the determinant

$$\Theta = \begin{vmatrix} \frac{\partial x_{\kappa 2}}{\partial u_\gamma} & 0 & P_{\rho\rho'} & 0 & 0 \\ \frac{\partial y_{\kappa 2}}{\partial u_\gamma} & 0 & P_{\kappa\rho'} & 0 & 0 \\ \frac{\partial v_{\kappa 2}}{\partial u_\gamma} & \frac{\partial v_{\kappa 2}}{\partial u_\lambda} & 0 & 0 & \delta_{i\kappa} \\ \frac{\partial l_\rho}{\partial u_\gamma} & \frac{\partial l_\rho}{\partial u_\lambda} & 0 & Q_\rho & 0 \end{vmatrix} \qquad (\rho' = 1, \cdots, r),$$

Where $P_{\rho\rho'}, P_{\kappa\rho'}, Q_\rho$ are as yet arbitrary matrices. They can be so chosen as to make $\Theta \neq 0$, since the first n columns are already of rank n. The product is

$$\Gamma = \begin{vmatrix} \frac{\partial \mathcal{Y}_i}{\partial u_\gamma} & \frac{\partial \mathcal{Y}_i}{\partial u_\lambda} & F_{i\rho'} & 0 & \frac{\partial \mathcal{Y}_i}{\partial v_{\kappa 2}} \\ C_{\rho\gamma} & 0 & G_{\rho\rho'} & 0 & 0 \\ D_\gamma & D_\lambda & H_{\rho'} & I & 0 \\ E_{i\gamma} & E_{i\lambda} & J_{i\rho'} & K_i & \delta_{i\kappa} \end{vmatrix},$$

where the elements $C_{\rho\gamma}, D_\gamma, D_\lambda, E_{i\gamma}, E_{i\lambda}$ all vanish, being the derivatives with respect to u_κ of the identities in u_κ, $f_\rho = 0$ and (8.15), and linear combinations of them. The values of $F_{i\rho'}, J_{i\rho'}, K_i, H_{\rho'}$ are immaterial. The n-rowed minor in the upper left hand corner is D, the $\partial \mathcal{Y}_i / \partial u_\gamma, \partial \mathcal{Y}_i / \partial u_\lambda$ being the same as in (8.21). Hence

$$\Gamma = D \cdot \Pi,$$

where

$$\Pi = \begin{vmatrix} G_{\rho\rho'} & 0 \\ H_{\rho'} & I \end{vmatrix} = I \cdot \left| G_{\rho\rho'} \right|.$$

Now the matrix $\| G_{\rho\rho'} \|$ is equal to the product of the matrices $\| \partial f_\rho / \partial x_2 \;\; \partial f_\rho / \partial y_{i2} \|$ and $\left\| \begin{matrix} P_{o\rho'} \\ P_{i\rho'} \end{matrix} \right\|$. The former is of rank r, so evidently there exists a choice of the P's making the product have the same rank. The element I is equal to $(\partial f_\rho / \partial x_2 + y_i'(x_2)\, \partial f_\rho / \partial y_{i2}) Q_\rho$ and hence is different from zero for a proper choice of the Q's, provided $\partial f_\rho / \partial x_2 + y_i'(x_2)\, \partial f_\rho / \partial y_{i2}$ are not all zero. Thus Π can be chosen different from zero. Now Π and Θ are independent of x. If $D \not\equiv 0$ on $x_1 x_2$ then the relation $\Delta\Theta = D\Pi$ implies that $\Theta \neq 0$. Or, if $\Delta \not\equiv 0$, Θ can be chosen different from zero, which implies $\Pi \neq 0$. Hence the theorem:

THEOREM. *If E_{12} is a normal extremal which satisfies the transversality conditions and has $R \neq 0$ and either D or Δ not identically zero along it, and if $\partial f_\rho / \partial x_2 + y_i'(x_2)\, \partial f_\rho / \partial y_{i2}$ are not all zero for E_{12}, then the zeros of D and Δ are the same for E_{12}.*

VITA

Moffatt Grier Boyce was born February 3, 1900, near Atoka, Tennessee, and received his early schooling there. After four years in Erskine College, Due West, South Carolina, he was awarded the Bachelor of Arts degree in 1922. For the next three years he taught mathematics in Bryson College, Fayetteville, Tennessee, and did further work of undergraduate rank at the University of Chicago in the summers. The two full years, 1925-1927, and the summers of 1928, 1929, were then spent in graduate study at the latter institution. In 1926 he received the Master of Science degree. Since 1927 he has been an instructor in Adelbert College of Western Reserve University. During his residence at the University of Chicago it has been his privilege to study under Professors Moore, Slaught, Dickson, Bliss, Moulton, MacMillan, Lane, Graves, Barnard and others. To all these he feels indebted, but especially to Professors Bliss and Graves is he grateful for their many suggestions and kindly interest in the preparation of this thesis.